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Remark on an Algebraic Solution of Painlevé Sixth Equation Obtained by Hitchin

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1 Introduction

The purpose of this paper is to study a free divisor related with an algeberaic solution of Painlevé sixth equation found by Hitchin [2]. Applying M. Kato's idea, we construct an algebraic solution of Painlevé sixth equation from a holonomic system of rank two associated with a free divisor in $\mathbf{C}^3$ defined by a weighted homogeneous polynomial.

2 A free divisor related with a polynomial of icosahedral coefficients

We start this note with introducing a matrix with polynomial entries

$$M = \begin{pmatrix} x_1 & 2x_2 & 4x_3 \\ -135x_1(x_1^2+3x_2) & -x_1^4+324x_1^2x_2+81x_2^2+x_3 & 108(19x_1^6+741x_1^4x_2-4617x_1^2x_2^2+243x_2^3) \\ x_1^4-324x_1^2x_2-81x_2^2+x_3 & -20x_1x_2(x_1^2-27x_2) & 8x_1(x_1^6+1539x_1^4x_2+20007x_1^2x_2^2-41553x_2^3) \end{pmatrix}$$

Using M, we define $h_0 = -\frac{1}{4}\det M$. Then h_0 is regarded as a cubic polynomial of x_3:

$$h_0 = x_3^3 + \sigma_1 x_3^2 + \sigma_2 x_3 + \sigma_3,$$

where

$$\begin{aligned} \sigma_1 &= 0, \\ \sigma_2 &= -3(x_1^8+2052x_1^6x_2+40014x_1^4x_2^2-166212x_1^2x_2^3+6561x_2^4), \\ \sigma_3 &= 2(x_1^4+81x_2^2)(x_1^8-4698x_1^6x_2-810486x_1^4x_2^2+380538x_1^2x_2^3+6561x_2^4). \end{aligned}$$

Let V_1, V_2, V_3 be vector fields defined by the matrix M, namely,

$${}^t(V_1, V_2, V_3) = M\,{}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

Then it is easy to see that V_1 is an Euler vector field with weight $(1, 2, 4)$ and that

$$V_1 h_0 = 12h_0, \quad V_2 h_0 = V_3 h_0 = 0.$$

Moreover

$$[V_1, V_2] = 2V_2, \quad [V_1, V_3] = 3V_3,$$

and

$$[V_2, V_3] = -16x_1(x_1^2 - 27x_2)V_2 + 81(x_1^2+3x_2)V_3.$$

As a consequence, $h_0 = 0$ is a free divisor in $\mathbf{C}^3$ in the sense of Saito [4].

The j-invariant of h_0 is defined to be

$$J(\sigma_1, \sigma_2, \sigma_3) = \frac{(\sigma_1^2 - 3\sigma_2)^3}{-27(\sigma_3 - \frac{1}{3}\sigma_1\sigma_2 + \frac{2}{27}\sigma_1^3)^2 - 4(\sigma_2 - \frac{1}{3}\sigma_1^2)^3} = \frac{27\sigma_2^3}{27\sigma_3^2 + 4\sigma_2^3}.$$

Put $m = \dfrac{x_2}{x_1^2}$. Then $J(\sigma_1, \sigma_2, \sigma_3)$ turns out to be

$$\varphi(m) = -\frac{(1+2052m+40014m^2-166212m^3+6561m^4)^3}{2304m(-1+99m+81m^2)^5}.$$

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Remark 2.1 *Putting* $m = \frac{z_1^5}{9z_2^5}$, *we find that* $\varphi(m)$ *takes the form* $\frac{H^3}{256f^5}$, *where*

$$\begin{array}{rcl} f &=& z_1z_2(z_1^{10}+11z_1^5z_2^5-z_2^{10}), \\ H &=& -(z_1^{20}+z_2^{20})+228(z_1^{15}z_2^5-z_1^5z_2^{15})-494z_1^{10}z_2^{10}. \end{array}$$

As is shown in Klein's book [3], f *and* H *are the defining polynomial of vertices and that of centers of faces of Icosahedron, respectively. It is interesting to explain the reason why the polynomials* f *and* T *appear in the numerator and the denominator of* $\varphi(m)$.

3 Solution of $J(t) = \varphi(m)$

The j-invariant of the cubic polynomial $P_1(z) = z(z-1)(z-t)$ is known to be

$$J(t) = \frac{(1-t+t^2)^3}{t^2(1-t)^2}.$$

It is straightforward to show that

$$\begin{array}{rcl} t &=& \dfrac{1+2s-5s^2-5s^4-2s^5+s^6-8s^2u}{1+2s-5s^2-5s^4-2s^5+s^6+8s^2u} \\ m &=& \dfrac{(-11+5\sqrt{5})(-2+s-\sqrt{5})^2(1+2s-\sqrt{5})}{18(-2+s+\sqrt{5})^2(1+2s+\sqrt{5})} \end{array} \qquad (1)$$

give a solution of

$$J(t) = \varphi(m),$$

where $u^2 = s(s^2+s-1)$. If $x_2 = mx_1^2$, then

$$\begin{array}{rcl} \sigma_2 &=& -3(1+2052m+40014m^2-166212m^3+6561m^4)x_1^8, \\ \sigma_3 &=& 2(1+81m^2)(1-4698m-810486m^2+380538m^3+6561m^4)x_1^{12}. \end{array}$$

Moreover, by substituting m with the rational function of s in (1), we find that

$$\begin{array}{rcl} \sigma_2 &=& -\frac{3\cdot 5^4(-25+11\sqrt{5})^2(1+4s-6s^2-20s^3+15s^4-216s^5+236s^6+216s^7+15s^8+20s^9-6s^{10}-4s^{11}+s^{12})x_1^8}{2^2(s-2+\sqrt{5})^8(s+(1+\sqrt{5})/2)^4}, \\ \sigma_3 &=& \frac{5^6(-25+11\sqrt{5})^3(1+s^2)(1+2s-6s^2-2s^3+s^4)(1-4s+6s^2+4s^3+s^4)(1+8s+20s^2+8s^3-106s^4-8s^5+20s^6-8s^7+s^8)x_1^{12}}{2^2(s-2+\sqrt{5})^{12}(s+(1+\sqrt{5})/2)^6}. \end{array}$$

We put $P(z) = P_1(z+\frac{t+1}{3})(=(z+\frac{t+1}{3})(z+\frac{t-2}{3})(z+\frac{1-2t}{3}))$ and introduce polynomials τ_2, τ_3 of t by

$$P(z) = z^3+\tau_2z+\tau_3.$$

Clearly the equation $P(z) = 0$ has three roots

$$\zeta_0 = \frac{-t-1}{3}, \quad \zeta_1 = \frac{2-t}{3}, \quad \zeta_2 = \frac{2t-1}{3}.$$

We regard t as a rational function of s by the relation in (1) for a moment. Then it is straightforward to show that

$$\begin{array}{rcl} \tau_2 &=& -\frac{(1+4s-6s^2-20s^3+15s^4-216s^5+236s^6+216s^7+15s^8+20s^9-6s^{10}-4s^{11}+s^{12})}{3(1+2s-5s^2-5s^4-2s^5+s^6+8s^2u)^2}, \\ \tau_3 &=& \frac{2(1+s^2)(1+2s-6s^2-2s^3+s^4)(1-4s+6s^2+4s^3+s^4)(1+8s+20s^2+8s^3-106s^4-8s^5+20s^6-8s^7+s^8)}{27(1+2s-5s^2-5s^4-2s^5+s^6+8s^2u)^3}. \end{array}$$

Then computing the ratio $\frac{\sigma_j}{\tau_j}$ $(j = 2, 3)$, we find that

$$\sigma_2 = Q(s)^2\tau_2, \quad \sigma_3 = Q(s)^3\tau_3,$$

where

$$Q(s) = \frac{75(-25+11\sqrt{5})(1+2s-5s^2-5s^4-2s^5+s^6+8s^2u)x_1^4}{2(s-2+\sqrt{5})^4(s+(1+\sqrt{5})/2)^2}$$

As a consequence, we find that

$$h_0 = x_3^3+\sigma_2x_3+\sigma_3 = (x_3-Q(s)\zeta_0)(x_3-Q(s)\zeta_1)(x_3-Q(s)\zeta_2).$$

Noting this, we put $z_j = Q(s)\zeta_j$ $(j = 0, 1, 2)$. Then $h_0 = (x_3-z_0)(x_3-z_1)(x_3-z_2)$.

4 Holonomic system related with the free divisor $h_0 = 0$

We introduce the following matrices to construct a holonomic system.

$$\begin{aligned} A_1 &= \begin{pmatrix} r_0 & 0 \\ 0 & 2+r_0 \end{pmatrix} \\ A_2 &= \begin{pmatrix} 0 & 1 \\ -162\{(-107+72\sqrt{5})x_1^4+162(4+3\sqrt{5})x_1^2x_2-243x_2^2-x_3\} & 0 \end{pmatrix} \\ A_3 &= \begin{pmatrix} 8x_1\{(-7+3\sqrt{5})x_1^2+9(1+\sqrt{5})x_2\} & \frac{2}{9}(-2+\sqrt{5})x_1 \\ -108x_1\begin{pmatrix} 5(-12+5\sqrt{5})x_1^4+18(-25+8\sqrt{5})x_1^2x_2 \\ +27(20+11\sqrt{5})x_2^2-\sqrt{5}x_3 \end{pmatrix} & -8x_1\{(-7+3\sqrt{5})x_1^2+9(1+\sqrt{5})x_2\} \end{pmatrix} \end{aligned}$$

Then

$$V_j\vec{u} = A_j\vec{u}\ (j=1,2,3) \tag{2}$$

is a holonomic system, where $\vec{u} = \begin{pmatrix} u \\ V_2u \end{pmatrix}$ in this case. The system (2) is equivalent to

$$\partial_{x_j}\vec{u} = B_j\vec{u}\ (j=1,2,3) \tag{3}$$

where B_1, B_2, B_3 are 2×2 matrices such that each of h_0B_j has polynomial entries. The equation $\partial_{x_3}\vec{u} = B_3\vec{u}$ is rewritten as a differential equation for u:

$$\partial_{x_3}^2 u + \frac{\tilde{P}_1(x)}{(x_3-a_s)h_0}\partial_{x_3}u + \frac{\tilde{P}_2(x)}{(x_3-a_s)h_0^2}u = 0 \tag{4}$$

where both $\tilde{P}_1, \tilde{P}_2$ are polynomials of x_3 and

$$a_s = \frac{(-2+\sqrt{5})x_1^6+(1089-594\sqrt{5})x_1^4x_2+(7128-891\sqrt{5})x_1^2x_2^2+729x_2^3}{(-2+\sqrt{5})x_1^2+9x_2}.$$

Now we put $u = h_0^{p_1}v$ for a constant p_1 and obtain a differential equation for v from (4). In this case, if $p_1 = \frac{r_0-2}{12}$, then the equation for v becomes of the form

$$\partial_{x_3}^2 v + \left\{(1-r)\frac{h_0'}{h_0} - \frac{1}{x_3-a_s}\right\}\partial_{x_3}v + \left\{\frac{c_0x_3^2+c_1x_3+c_2}{h_0} - \frac{c_0}{x_3-a_s}\right\}v = 0, \tag{5}$$

where r is a constant, c_0, c_1, c_2 are rational functions of x_1, x_2 and $h_0' = \partial_{x_3}h_0$. Equation(5) is regarded as an ordinary differential equation with respect to the variable x_3.

Since $h_0 = (x_3-z_0)(x_3-z_1)(x_3-z_2)$, there are four singular points $x_3 = z_0, z_1, z_2, a_s$ of (5). By the coordinate transformation $w = \dfrac{x_3-z_0}{z_1-z_0}$, the points z_0, z_1, z_2 are transformed to $0, 1,\ t = \dfrac{z_2-z_0}{z_1-z_0}$, respectively. We now put $\lambda = \dfrac{a_s-z_0}{z_1-z_0}$. Then Equation (5) turns out to be

$$\tilde{v}'' + \left\{(1-r)\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right) - \frac{1}{w-\lambda}\right\}\tilde{v}' + \left(\frac{A}{w}+\frac{B}{w-1}+\frac{C}{w-t}+\frac{D}{w-\lambda}\right)\tilde{v} = 0, \tag{6}$$

where $\tilde{v}$ is a function of w, $\tilde{v}' = \dfrac{d}{dw}\tilde{v}, \tilde{v}'' = \dfrac{d^2}{dw^2}\tilde{v}$ and A, B, C, D are independent of w. It is underlined here that λ is regarded as a function of t. This follows from the computation in the previous section.

5 Relationship with Hitchin's solution

Since $\lambda = \frac{a_s-z_0}{z_1-z_0}$, we obtain the parametric expression of λ :

$$\lambda = \frac{(-1+s)^4(1+s)^2(3+s)(-1-4s+s^2)}{(-3+7s-s^2+s^3)(1+2s-5s^2-5s^4-2s^5+s^6+8s^2u)}. \tag{7}$$

We now recall the elliptic dihedral solution which is one of algebraic solutions of Painlevé sixth equation given in [1], p.11:

$$t_H = \frac{(s^2+u)^2(s(s+2)-u)(s(s-2)-u)}{(s^2-u)^2(s(s+2)+u)(s(s-2)+u)}, \quad y = \frac{(3s-1)(s^2-4s-1)(s^2+u)(s(s+2)-u)}{(3s^3+7s^2+s+1)(s^2-u)(s(s-2)+u)}, \tag{8}$$

where (s,u) lives on the elliptic curve $u^2 = s(s^2+s-1)$. Note that this solution is obtained by Hitchin [2]. For our purpose, we rewrite t_H and y by using the relation between s and u. Then

$$t_H = \frac{1+2s-5s^2-5s^4-2s^5+s^6-8s^2u}{1+2s-5s^2-5s^4-2s^5+s^6+8s^2u}, \; y = \frac{(-1+3s)(-1-4s+s^2)(1-s+s^2+s^3+2u)}{(1+s+7s^2+3s^3)(1-s-3s^2+s^3+2u)}. \tag{9}$$

Clearly t_H coincides with the function t defined in (1) but y seems not to equal to the function λ defined in (7). To explain the difference, we first point out the existence of an automorphism χ on the elliptic curve $u^2 = s(s^2+s-1)$ defined by $(s,u) \to (-\frac{1}{s}, \frac{u}{s^2})$. By the action of χ, t_H is fixed but y turns out to be

$$y_H = \frac{(s+3)(-1-4s+s^2)(-1+s+s^2+s^3+2su)}{(s^3-s^2+7s-3)(s^3+s^2-3s-1+2su)}.$$

By direct computation, we find that y_H coincides with the function λ of (7). As a consequence, (t,λ) is equivalent to the solution obtained by Hitchin. At last we note that $\theta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in this case (cf.[1]).

References

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